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Oscillation of third order damped nonlinear differential equation

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Abstract: The purpose of this paper is to give oscillation criteria for the third order nonlinear differential equation with daming term

$$[a_2(t)\{(a_1(t)x'(t)'\}]' + p(t)x'(t) + \sum_{i=1}^{n} q_i(t)f(x(g_i(t))) = 0,$$

by using Riccati trasformatiom teqnique and comparison with first order differential equation whose oscillatory characters are known. Our results generalize and improve some known results for oscillation of third order nonlinear differential equations. Some examples are given to illustrate the main results.

Keywords: Oscillation; third order; differential equations

1. Introduction

In this paper, we are concerned with the oscillation of third order nonlinear differential equation with damping term

$$[a_2(t)\{(a_1(t)x'(t)'\}]' + p(t)x'(t) + \sum_{i=1}^{n} q_i(t)f(x(g_i(t))) = 0, \tag{1}$$

where the following conditions are satisfied

(A1) $a_1(t)$, $a_2(t)$ $p(t)$ and $q(t)$ $\in C([t_0, \infty), (0, \infty))$;
(A2) $f \in C(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$, $f'(x) > 0$ for all $x \neq 0$ and $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$;
(A3) $g(t) \in C^1([t_0, \infty), \mathbb{R})$ for $t \in [t_0, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$.

We mean by a solution of equation (1) a function $x(t) : [t_x, \infty) \to \mathbb{R}$, $t_x \geq t_0$ such that $x(t)$, $a_1(t)(x'(t))^{\alpha_1}$, $a_2(t)\{(a_1(t)(x'(t))^{\alpha_1})'\}^{\alpha_2}$ are continuously differentiable for all $t \in [t_x, \infty)$ and satisfies (1) for all $t \in [t_x, \infty)$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. A solution of equation (1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. In the sequel it will be always assumed that equation (1) has nontrivial solutions which exist for all

$t_0 \geq 0$. Equation (1) is called oscillatory if all solutions are oscillatory. In fact, Tiryaki and Aktas [20] studied the oscillation of third order nonlinear differential equation with damping term of the form

$$\left(a_2(t)\left[a_1(t)x'(t)\right]'\right)' + p(t)x'(t) + q(t)f(x(g(t))) = 0, \tag{2}$$

under the condition $g(t) \leq t$. Aktas *et al* [6,7] established some sufficient conditions for the third order nonlinear differential equations with damping term

$$\left(a_2(t)\left[a_1(t)x'(t)\right]'\right)' + p(t)x'(t) + q(t)f(x(t)) = 0,$$

and (2) without the condition $g(t) \leq t$. A number of sufficient conditions for oscillation were obtained , for $k = 1, 2$

$$\int_{t_0}^{\infty} a_k^{-1}(t)dt = \infty, \tag{3}$$

Therefore it will be great interest to estabilsh oscillation criteria for equation (1) for both of the cases (3) and

$$\int_{t_0}^{\infty} a_k^{-1}(t)dt < \infty. \tag{4}$$

By using Riccati transformation technique and a comparison with some first order differential equation whose oscillatory characters are known. Our results will improve and extend results in [7,20] and many known results.

2. Main Results

Before stating our main results, we start with the following lemmas which will play an important role in the proofs of our main results. We let,

$$\begin{aligned} \delta(t, t_0) \quad &: \quad = \int_{t_0}^{t} a_1^{-1}(v)dv, \ \delta_k(t) := \int_{t}^{\infty} a_k^{-1}(v)dv, \ k = 1, 2 \\ \beta(t, t_0) \quad &: \quad = \int_{t_0}^{t} a_2^{-1}(s)ds, \ g(t) := \min(g_1(t), g_2(t), ..., g_n(t)) \end{aligned}$$

Lemma 1. *Suppose that*

$$[a_2(t)\left(z'(t)\right)]' + \frac{p(t)}{a_1(t)}z(t) = 0 \tag{5}$$

is nonoscillatory. If x is a nonoscillatory solution of (1) *on $[T, \infty), T \geq t_0$, then there exists a $t_0 \in [T, \infty)$ such that either $x(t)x'(t) > 0$ or $x(t)x'(t) < 0$ for $t \geq t_0$.*

Proof. Suppose that equation (1) has a nonoscillatory solution x on $[t_0, \infty)$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)$, sufficiently large such that $x(t) > 0$ and $x(g(t)) > 0$ on $[t_1, \infty)$. Clearly, $y(t) := -a_1(t)\left(x'(t)\right)^{\alpha_1}$ is a solution of the second order nonhomogeneous differential equation

$$[a_2(t)\left(y'(t)\right)]' + \frac{p(t)}{a_1(t)}y(t) = \sum_{i=1}^{n} q_i(t)f(x(g_i(t))), \tag{6}$$

We claim that the solution (6) are nonoscillatory. Suppose not, let y is oscillatory solution (6) with consecutive zeros at b and c $(t_1 < b < c)$ such that $y'(b) \geq 0$ and $y'(c) \leq 0$. Let z be a solution of (5). Multiply (6) by $z(t)$ and using (5), we obtain

$$z(t)[a_2(t)\left(y'(t)\right)]' - [a_2(t)\left(z'(t)\right)]'y(t) = \sum_{i=1}^{n} z(t)q_i(t)f(x(g_i(t))),$$

It can be written as follows

$$\left(a_2(t)z(t)y'(t) - a_2(t)z'(t)y(t)\right)' = \sum_{i=1}^{n} z(t)q_i(t)f(x(g_i(t))),$$

Integrating the above inequality from b to c, we get a contradiction. The proof is complete. □

Lemma 2. *Assume that*
(H1) Either

$$\int_{t_0}^{\infty} a_2^{-1}(t)dt = \infty, \tag{7}$$

or

$$\int_{t_0}^{\infty} \left(a_2^{-1}(s)\left(\sum_{i=1}^{n}\int_{t_0}^{s}(q_*(r)+p_*(r))\,dr\right)\right)ds = \infty, \tag{8}$$

where

$$q_*(t) := q_i(t)f(\delta_2(g(t)))f(\delta(g(t),t_2)),$$

and

$$p_*(t) := p(t)\delta_2(t)a_1^{-1}(t),$$

hold for $g(t) \geq T$. Let $x(t)$ be an eventually positive solution of the equation (1) such that $x'(t) > 0$. Then there exists a $T \geq t_0$ such that

$$\left(a_1(t)\left(x'(t)\right)\right)' > 0 \text{ and } [a_2(t)\{(a_1(t)\left(x'(t)\right))'\}]' < 0.$$

Proof. Pick $t_1 \geq t_0$ such that $x(g(t)) > 0$, for $t \geq t_1$. Since $x(t)$ is an eventually positive solution of the equation (1) such that $x'(t) > 0$ for all $t \in [t_0, \infty)$. From equation (1), (A1) and (A3), we have

$$[a_2(t)\{(a_1(t)\left(x'(t)\right))'\}]' < 0,$$

for all $t \geq t_1$. Then $a_2(t)\,(a_1(t)\,(x'(t)))'$ is strictly decreasing on $[t_1, \infty)$, so either $(a_1(t)\,(x'(t)))' > 0$ or $(a_1(t)\,(x'(t)))' < 0$. We claim that $(a_1(t)\,(x'(t)))' > 0$ on $[t_1, \infty)$. If not, then, we have, $a_1(t)\,(x'(t))$is strictly decreasing on $[t_2, \infty)$ and there exists a negative constant M such that

$$a_2(t)\{(a_1(t)(x'(t)))'\} < M \text{ for all } t \geq t_2.$$

Dividing by $a_2(t)$ and integrating from t_2 to t

$$a_1(t)(x'(t)) \leq a_1(t_2)\left(x'(t_2)\right) + M^{\frac{1}{\alpha_2}}\int_{t_2}^{t} a_2^{-1}(s)ds.$$

Letting $t \to \infty$, and using (7) then $a_1(t)\,(x'(t)) \to -\infty$, which contradicts that $x'(t) > 0$. Otherwise, if (8) is satisfied, we have

$$\begin{aligned}
x(t) - x(t_3) &= \int_{t_3}^{t} x'(u)\,du \\
&= \int_{t_3}^{t} a_1^{-1}(u)\left(a_1(u)\left(x'(u)\right)\right)du \\
&\geq \left(a_1(t)\left(x'(t)\right)\right)\int_{t_3}^{t} a_1^{-1}(u)du, \quad \text{for } t \geq t_3,
\end{aligned}$$

and hence

$$x(t) \geq \left(a_1(t)\left(x'(t)\right)\right)\int_{t_3}^{t} a_1^{-1}(u)du \quad \text{for } t \geq t_3.$$

There exists a $t_4 \geq t_3$ with $g(t) \geq t_3$ for all $t \geq t_4$ such that

$$x(g(t)) \geq y(g(t))\delta(g(t), t_3) \quad \text{for } t \geq t_4.$$

where $y(t) := a_1(t)\,(x'(t))$. It is clear that $y(t) > 0$ and $y'(t) < 0$. It follows that

$$-a_2(t)(y'(t)) \geq -a_2(t_4)(y'(t_4)) \text{ for } t \geq t_4,$$

thus

$$-y'(t) \geq -\frac{a_2(t_4)y'(t_4)}{a_2(t)} \quad \text{for} \quad t \geq t_4.$$

Integrate the above inequality from t to ∞, we get

$$y(t) \geq -a_2(t_4)y'(t_4)\delta_2(t),$$

then,

$$y(t) \geq k_1\delta_2(t), \quad \text{for} \quad t \geq t_4 \tag{9}$$

where $k_1 := -a_2(t_4)y'(t_4) > 0$. There exists a $t_5 \geq t_4$ with $g(t) \geq t_4$ for all $t \geq t_5$ such that

$$y(g(t)) \geq k_1\delta_2(g(t)) \quad \text{for all} \quad t \geq t_5. \tag{10}$$

The inequality (9) yields

$$a_1(t)\left(x'(t)\right) \geq k_1\delta_2(t),$$

that is

$$x'(t) \geq (k_1\delta_2(t))a_1^{-1}(t).$$

From Eq.(1), (10), (A2) and the above inequality, we get, for $t \geq t_4$,

$$\begin{aligned} 0 &= \left(a_2(t)(y'(t))\right)' + p(t)x'(t) + \sum_{i=1}^{n} q_i(t)f(x(g_i(t))) \\ &\geq (a_2(t)(y'(t)))' + p(t)x'(t) + f(x(g(t)))\sum_{i=1}^{n} q_i(t) \\ &\geq \left(a_2(t)(y'(t))\right)' + p(t)(k_1\delta_2(t))a_1^{-1}(t) \\ &\quad + f(y(g(t)))f\left(\delta(g(t), t_3)\right)\sum_{i=1}^{n} q_i(t). \end{aligned} \tag{11}$$

By integrating the above inequality from t_5 to t, we get

$$\begin{aligned} &\sum_{i=1}^{n}\int_{t_5}^{t}[q_i(r)f(k_1\delta_2(g(r)))f(\delta(g(r), t_3))) + p(r)(k_1\delta_2(r))a_1^{-1}(r)]dr \\ \leq\ & a_2(t_5)(y'(t_5)) - a_2(t)(y'(t)), \end{aligned}$$

Using (A2), we obtain

$$\begin{aligned} & a_2^{-1}(t)\left(\sum_{i=1}^{n}\int_{t_5}^{t}\left[bq_*(r) + k_1p_*(r)\right]dr\right) \\ \leq\ & -y'(t), \end{aligned}$$

where $b := f(k_1)$. Integrating the above inequality from t_5 to ∞, we get

$$\begin{aligned} & \int_{t_5}^{\infty}\left(a_2^{-1}(s)\left[\sum_{i=1}^{n}\int_{t_5}^{s}(bq_*(r) + k_1p_*(r))\,dr\right]\right)ds \\ \leq\ & y(t_5) < \infty, \end{aligned}$$

which contradicts the condition (8). The proof is complete. □

Lemma 3. *Assume that $p'(t) \leq 0$ hold. Let $x(t)$ is an eventually positive solution of the equation (1) for all $t \in [t_0, \infty)$ such that $x'(t) < 0$. If*

$$\sum_{i=1}^{n} \lim_{t\to\infty} \int_{t_0}^{t} q_i(s)ds = \infty, \tag{12}$$

and (H2) Either

$$\int_{t_0}^{\infty} a_1^{-1}(t)dt = \infty, \tag{13}$$

or

$$\int_{t_0}^{\infty} a_1^{-1}(u) \left(\int_{t_2}^{u} a_2^{-1}(s)ds \right) du = \infty, \tag{14}$$

are satisfied. Then $x(t) \to 0$ as $t \to \infty$.

Proof. Pick $t_1 \geq t_0$ such that $x(g_i(t)) > 0,\ (i = 1, 2, ..., n)$ for $t \geq t_1$. Since $x(t)$ is positive decreasing solution of the equation (1) then, we get, $\lim_{t\to\infty} x(t) = l_1 \geq 0$. Assume $l_1 > 0$, then, $x(g_i(t)) \geq l_1$ $(i = 1, 2, ..., n)$ for $t \geq t_2 \geq t_1$. Integrating equation (1) from t_1 to t, we find

$$a_2(t)\{(a_1(t)\left(x'(t)\right))'\} \leq c - x(t)p(t) - \sum_{i=1}^{n} \int_{t_1}^{t} \left[q_i(s)f(x(g_i(s))) - p'(s)x(s)\right] ds.$$

where $c := a_2(t_1)\{(a_1(t_1)\left(x'(t_1)\right))'\} + x(t_1)p(t_1)$. It follows that

$$a_2(t)\{(a_1(t)\left(x'(t)\right))'\} \leq c - f(l_1) \sum_{i=1}^{n} \int_{t_1}^{t} q_i(s)ds,$$

and hence

$$a_2(t)\{(a_1(t)\left(x'(t)\right))'\} \to -\infty \text{ as } t \to \infty. \tag{15}$$

So, $(a_1(t)\left(x'(t)\right))' < 0$ such that

$$a_1(t)\left(x'(t)\right) \leq a_1(t_2)\left(x'(t_2)\right) = k < 0.$$

Dividing by $a_1(t)$ and integrating from t_2 to t, we get

$$x(t) \leq x(t_2) + k \int_{t_2}^{t} a_1^{-1}(s)ds.$$

Letting $t \to \infty$, then (13) yields $x(t) \to -\infty$ this contradicts the fact that $x(t) > 0$. Otherwise, if (14) is satisfied. From (15), we have, for $A < 0$

$$a_2(t)\{(a_1(t)\left(x'(t)\right))'\} \leq A,$$

for sufficiently large t. Dividing by $a_2(t)$ and integrating the above inequality from t_2 to t, we obtain

$$a_1(t)\left(x'(t)\right) \leq A \int_{t_2}^{t} a_2^{-1}(s)ds,$$

Dividing by $a_1(t)$ and integrating from t_2 to t, we have

$$x(t) \leq x(t_2) + A \int_{t_2}^{t} a_1^{-1}(u) \left(\int_{t_2}^{u} a_2^{-1}(s)ds \right) du.$$

Let $t \to \infty$. From condition (14), we get $x(t) \to -\infty$ which contradicts the fact that $x(t) > 0$. Then $\lim_{t\to\infty} x(t) = 0$. □

Theorem 1. *Let (H1), (H2), $p'(t) \leq 0$,(12), $g'(t) > 0$ on $[t_0, \infty)$ hold and equation (5) is oscillatory. If the first order delay equations*

$$y'(t) + p_1(t)y(t) + q_1(t)f(y(g(t))) = 0, \tag{16}$$

is oscillatory where

$$p_1(t) := p(t)a_1^{-1}(t)\left[\int_{t_2}^{t} a_2^{-1}(s)ds\right],$$

and

$$q_1(t) := \sum_{i=1}^{n} q_i(t)f\left(\int_{t_0}^{g(t)} a_1^{-1}(s)\left[\int_{t_0}^{s} a_2^{-1}(u)du\right]ds\right),$$

then equation (1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Pick $t_1 \geq t_0$ such that $x(g_i(t)) > 0$, $(i = 1, 2, ..., n)$ for $t \geq t_1$. Since $x\ (t)$ is an eventually positive solution of the equation (1) for all $t \in [t_0, \infty)$. Then, from Lemma 1, it follows that $x'(t) > 0$ or $x'(t) < 0$. If $x'(t) > 0$ from Lemma 2, we have, $(a_1(t)\,(x'(t)))' > 0$ and $[a_2(t)\{(a_1(t)\,(x'(t)))'\}]' < 0$, for all $t \geq t_1$,then

$$\begin{aligned} a_1(t)\left(x'(t)\right) &= a_1(t_2)\left(x'(t_2)\right) + \int_{t_2}^{t} a_2^{-1}(s)y(s)ds \\ &\geq y(t)\int_{t_2}^{t} a_2^{-1}(s)ds, \end{aligned}$$

where $y(t) := a_2(t)\{(a_1(t)\,(x'(t)))'\}$. It follows that

$$x'(t) \geq a_1^{-1}(t)y(t)\left[\int_{t_2}^{t} a_2^{-1}(s)ds\right]. \tag{17}$$

Integrating the above inequality from t_2 to t, we get

$$\begin{aligned} x(t) &\geq \int_{t_2}^{t} a_1^{-1}(s)y(s)\left[\int_{t_2}^{s} a_2^{-1}(u)du\right]ds \\ &\geq y(t)\int_{t_2}^{t} a_1^{-1}(s)\left[\int_{t_2}^{s} a_2^{-1}(u)du\right]ds. \end{aligned}$$

There exists $t_3 \geq t_2$ such that $g(t) \geq t_2$ for all $t \geq t_3$. Then

$$x(g(t)) \geq y(g(t))\int_{t_2}^{g(t)} a_1^{-1}(s)\left[\int_{t_2}^{s} a_2^{-1}(u)du\right]ds, \text{ for all } t \geq t_3.$$

Thus equation (1) and (A2) yield, for all $t \geq t_3$

$$\begin{aligned} -y'(t) &= \sum_{i=1}^{n} q_i(t)f(x(g_i(t))) + p(t)x'(t) \geq \sum_{i=1}^{n} q_i(t)f(x(g(t))) + p(t)x'(t) \\ &\geq \sum_{i=1}^{n} q_i(t)f(y(g(t)))f\left(\int_{t_2}^{g(t)} a_1^{-1}(s)\left[\int_{t_2}^{s} a_2^{-1}(u)du\right]ds\right) \\ &\quad + p(t)a_1^{-1}(t)y(t)\left[\int_{t_2}^{t} a_2^{-1}(s)ds\right]. \end{aligned}$$

Integrating the above inequality from t to ∞, we get

$$\begin{aligned} y(t) &\geq \sum_{i=1}^{n}\int_{t}^{\infty} q_i(s)f(y(g(s)))f\left(\int_{t_2}^{g(s)} a_1^{-1}(v)\left[\int_{t_2}^{v} a_2^{-1}(u)du\right]dv\right)ds \\ &\quad + \int_{t}^{\infty} p(s)a_1^{-1}(s)y(s)\left[\int_{t_2}^{s} a_2^{-1}(u)du\right]ds. \end{aligned}$$

The function $y(t)$ is obviously strictly decreasing. Hence, by [19, Theorem 1] there exists a positive solution of equation (16) which tends to zero this contradicts that (16) is oscillatory. If $x'(t) < 0$ from Lemma 3, we get $x(t)$ tends to zero as $t \to \infty$. The proof is complete. □

The next results is obtained by using Riccati transformation technique.

Theorem 2. *Let $g(t) < t$, (H1), (H2), $p'(t) \leq 0$,(12), $\frac{f(x)}{x} \geq K > 0$ hold and equation (5) is oscillatory. Furthermore, assume that there exists a positive differentiable function $\rho \in C^1([t_0, \infty), \mathbb{R})$ for sufficiently large t_1 such that*

$$\limsup_{t\to\infty} \sum_{i=1}^{n} \int_{t_0}^{t} \left(q_i(s)\rho(t) - \frac{\left(\rho'(s) - p(s)\rho(s)a_1^{-1}(t)\beta_1(s,t_0)\right)^2}{4\rho(s)g'(s)a_1^{-1}(g(t))\beta_1(g(s),t_0)} \right) ds = \infty, \tag{18}$$

Then every solution of equation (1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Pick $t_1 \geq t_0$ such that $x(g_i(t)) > 0$, for $t \geq t_1$. Since $x\ (t)$ is an eventually positive solution of the equation (1) for all $t \in [t_0, \infty)$. Then, from Lemma 1, it follows that $x'(t) > 0$ or $x'(t) < 0$. If $x'(t) > 0$ from Lemma 2, we have, $(a_1(t)\,(x'(t)))' > 0$ and $[a_2(t)\{(a_1(t)\,(x'(t)))'\}]' < 0$, for all $t \geq t$. Define the function $w(t)$ by

$$w(t) := \rho(t)\frac{a_2(t)\{(a_1(t)\,(x'(t)))'\}}{x(g(t))}.$$

Then

$$\begin{aligned} w'(t) &= \frac{\rho(t)}{x(g(t))}[a_2(t)\{(a_1(t)\,(x'(t)))'\}]' \\ &\quad -a_2(t)\{(a_1(t)\,(x'(t)))'\}\frac{\rho(t)x'(g(t))}{x^2(g(t))}g'(t) \\ &\quad +\frac{a_2(t)\{(a_1(t)\,(x'(t)))'\}}{x(g(t))}\rho'(t). \end{aligned}$$

It follows from equation (1), $g(t) \leq g_i(t)$

$$\begin{aligned} w'(t) &= \frac{\rho(t)}{x(g(t))}[-\sum_{i=1}^{n} q_i(t)f(x(g_i(t))) - p(t)x'(t)] \\ &\quad -a_2(t)\{(a_1(t)\,(x'(t)))'\}\frac{\rho(t)x'(g(t))}{x^2(g(t))}g'(t) \\ &\quad +\frac{a_2(t)\{(a_1(t)\,(x'(t)))'\}}{x(g(t))}\rho'(t) \\ &\leq \frac{\rho(t)}{x(g(t))}[-\sum_{i=1}^{n} q_i(t)f(x(g(t))) - p(t)x'(t)] \\ &\quad -a_2(t)\{(a_1(t)\,(x'(t)))'\}\frac{\rho(t)x'(g(t))}{x^2(g(t))}g'(t) \\ &\quad +\frac{a_2(t)\{(a_1(t)\,(x'(t)))'\}}{x(g(t))}\rho'(t). \end{aligned}$$

From (17) there exists $t_3 \geq t_2$ with $g(t) \geq t_2$ for all $t \geq t_3$ such that

$$\begin{aligned} x'(g(t)) &\geq a_1^{-1}(g(t))\beta_1(g(t),t_2)y(g(t)) \\ &\geq a_1^{-1}(g(t))\beta_1(g(t),t_2)y(t), \end{aligned} \tag{19}$$

where $y(t) := a_2(t)\{(a_1(t)\,(x'(t)))'\}$, $y'(t) < 0$. From the above inequality and (17), we obtain

$$\begin{aligned} w'(t) &\le -K\sum_{i=1}^{n} q_i(t)\rho(t) - \frac{p(t)\rho(t)}{x(g(t))}a_1^{-1}(t)\beta_1(t,t_2)y(t) + \frac{y(t)}{x(g(t))}\rho'(t) \\ &\quad -\frac{1}{x^2(g(t))}y^2(t)a_1^{-1}(g(t))\beta_1(g(t),t_2)\rho(t)g'(t). \end{aligned}$$

Thus

$$w'(t) \le -K\sum_{i=1}^{n} q_i(t)\rho(t) + \alpha(t)w(t) - \beta(t)w^2(t),$$

where

$$\alpha(t) := \frac{\rho'(t)}{\rho(t)} - p(t)a_1^{-1}(t)\beta_1(t,t_2) \text{ and } \beta(t) := \frac{1}{\rho(t)}g'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2),$$

and hence

$$\begin{aligned} w'(t) &\le -K\sum_{i=1}^{n} q_i(t)\rho(t) + \frac{\left(\rho'(t) - p(t)\rho(t)a_1^{-1}(t)\beta_1(t,t_2)\right)^2}{4\rho(t)g'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2)} \\ &\quad - \left(\sqrt{\beta(t)}w(t) - \frac{\rho'(t) - p(t)\rho(t)a_1^{-1}(t)\beta_1(t,t_2)}{2\sqrt{\rho(t)g'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2)}}\right)^2. \end{aligned}$$

Thus

$$w'(t) \le -K\sum_{i=1}^{n} q_i(t)\rho(t) + \frac{\left(\rho'(t) - p(t)\rho(t)a_1^{-1}(t)\beta_1(t,t_2)\right)^2}{4\rho(t)g'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2)}.$$

Integrating the above inequality from t_2 to t, we have

$$w(t) \le w(t_2) - \sum_{i=1}^{n}\int_{t_2}^{t}\left(Kq_i(s)\rho(s) - \frac{\left(\rho'(s) - p(s)\rho(s)a_1^{-1}(t)\beta_1(s,t_2)\right)^2}{4\rho(s)g'(s)a_1^{-1}(g(t))\beta_1(g(s),t_2)}\right)ds.$$

Letting $t \to \infty$. By the condition (18), we get $w(t) \to -\infty$ which contradicts the fact that $w(t) > 0$. When $x'(t) < 0$. All conditions of Lemma 3 are satisfied. Then $x(t) \to 0$ as $t \to \infty$. The proof is complete. □

Let $n = 1$ Theorem 2 generalize and completes Tiryaki and Aktas [20, Theorem 1].

Theorem 3. *Let (H1), (H2), $p'(t) \le 0$,(12), $f'(x) \ge L > 0$ and equation (5) is oscillatory. Furthermore, assume that there exists a positive differentiable function $\rho \in C^1([t_0,\infty),\mathbb{R})$ for sufficiently large t_1 such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\sum_{i=1}^{n} q_i(s)\rho(s) - \frac{\left(\rho'(s) - p(s)\rho(s)a_1^{-1}(s)\beta_1(s,t_0)\right)^2}{4L\rho(s)g'(s)a_1^{-1}(g(s))\beta_1(g(s),t_0)}\right)ds = \infty, \tag{20}$$

Then every solution of equation (1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Pick $t_1 \ge t_0$ such that $x(g_i(t)) > 0$, $(i = 1,2,...,n)$ for $t \ge t_1$. Since $x\ (t)$ is an eventually positive solution of the equation (1) for all $t \in [t_0,\infty)$. Then, from Lemma 1, it follows that $x'(t) > 0$ or $x'(t) < 0$. If $x'(t) > 0$ from Lemma 2, we have, $(a_1(t)\,(x'(t)))' > 0$ and $[a_2(t)\{(a_1(t)\,(x'(t)))'\}]' < 0$, for all $t \ge t$. Define the function $w(t)$ by

$$w(t) := \rho(t)\frac{a_2(t)\{(a_1(t)\,(x'(t)))'\}}{f(x(g(t)))}.$$

Then

$$\begin{aligned} w'(t) &= \frac{\rho(t)}{f(x(g(t)))}[a_2(t)\{(a_1(t)\,(x'(t)))'\}]' \\ &-a_2(t)\{(a_1(t)\,(x'(t)))'\}\frac{\rho(t)f'(x(g(t))x'(g(t))}{f^2(x(g(t)))}g'(t) \\ &+\frac{a_2(t)\{(a_1(t)\,(x'(t)))'\}}{f(x(g(t)))}\rho'(t). \end{aligned}$$

It follows from equation (1), $g(t) \leq g_i(t)$

$$\begin{aligned} w'(t) &\leq \frac{\rho(t)}{f(x(g(t)))}[-\sum_{i=1}^{n} q_i(t)f(x(g(t))) - p(t)x'(t)] \\ &-a_2(t)\{(a_1(t)\,(x'(t)))'\}\frac{\rho(t)f'(x(g(t))x'(g(t))}{f^2(x(g(t)))}g'(t) \\ &+\frac{a_2(t)\{(a_1(t)\,(x'(t)))'\}}{f(x(g(t)))}\rho'(t). \end{aligned}$$

From (17) and (19), we obtain

$$w'(t) \leq -\sum_{i=1}^{n} q_i(t)\rho(t) + \alpha(t)w(t) - \beta_1(t)w^2(t),$$

where

$$\alpha(t) := \frac{\rho'(t)}{\rho(t)} - p(t)a_1^{-1}(t)\beta_1(t,t_2) \text{ and } \beta_1(t) := \frac{L}{\rho(t)}g'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2),$$

and hence

$$\begin{aligned} w'(t) &\leq -\sum_{i=1}^{n} q_i(t)\rho(t) + \frac{\left(\rho'(t) - p(t)\rho(t)a_1^{-1}(t)\beta_1(t,t_2)\right)^2}{4\rho(t)g'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2)} \\ &-\left(\sqrt{\beta(t)}w(t) - \frac{\left(\rho'(t) - p(t)\rho(t)a_1^{-1}(t)\beta_1(t,t_2)\right)}{2\sqrt{\rho(t)Lg'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2)}}\right)^2. \end{aligned}$$

Thus

$$w'(t) \leq -\sum_{i=1}^{n} q_i(t)\rho(t) + \frac{\left(\rho'(t) - p(t)\rho(t)a_1^{-1}(t)\beta_1(t,t_2)\right)^2}{4L\rho(t)g'(t)a_1^{-1}(g(t))\beta_1(g(t),t_2)}.$$

Integrating the above inequality from t_2 to t, we have

$$w(t) \leq w(t_2) - \int_{t_2}^{t} (\sum_{i=1}^{n} q_i(s)\rho(s) - \frac{\left(\rho'(s) - p(s)\rho(s)a_1^{-1}(t)\beta_1(s,t_2)\right)^2}{4L\rho(s)g'(s)a_1^{-1}(g(t))\beta_1(g(s),t_2)})ds.$$

Letting $t \to \infty$. By the condition (20), we get $w(t) \to -\infty$ which contradicts the fact that $w(t) > 0$. When $x'(t) < 0$. All conditions of Lemma 3 are satisfied. Then $x(t) \to 0$ as $t \to \infty$. The proof is complete. □

Let $n = 1$ Theorem 3 generalize and improve Aktas *et al* [7, Theorem 1].

Example 1. *Consider the third order delay damped differential equation*

$$\left(\frac{1}{t}x'(t)\right)'' + \frac{1}{4t^3}x'(t) + e^t(x(\frac{t}{3})) + 3e^t(x(\frac{2t}{3})) = 0, \ t \geq 1. \tag{21}$$

We note that

$$f(x) = x,\ a_1(t) = \frac{1}{t},\ a_2(t) = 1,\ q_i := (2i-1)e^t,$$
$$g_i(t) = \frac{it}{3} < t,\ i = 1,2,\ g'(t) > 0,\ \lim_{t\to\infty} g_i(t) = \infty,$$

and

$$\int_1^\infty a_1^{-1}(u)du = \infty,\ \int_1^\infty a_2^{-1}(u)du = \infty.$$

we note that

$$z''(t) + \frac{1}{4t^2}z(t) = 0 \quad (22)$$

is nonoscillatory and it easy to see that (12) *and* (18)*hold. Then all conditions of Theorem* 2 *is satisfied then every nonoscillatory solution of Eq.*(21) *tends to zero as* $t \to \infty$.

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