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Oscillation criteria for third order nonlinear neutral differential equation

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Abstract: The purpose of this paper is to give oscillation criteria for the third order nonlinear neutral differential equation

$$[a_2(t)\{(a_1(t)((x(t)+p(t)x(\tau(t)))')^{\alpha_1})'\}^{\alpha_2}]' + q(t)f(x(g(t))) = 0.$$

Via comparison with some first order differential equations whose oscillatory characters are known. Our results generalize and improve some known results for oscillation of third order nonlinear differential equations. Some examples are given to illustrate our results.

Keywords: oscillation; third order; neutral differential equation.

1. Introduction

In this paper, we are concerned with the oscillation of third order nonlinear differential equation

$$[a_2(t)\{[a_1(t)\left(z'(t)\right)^{\alpha_1}]'\}^{\alpha_2}]' + q(t)f(x(g(t))) = 0, \tag{1}$$

where $z(t) := x(t) + p(t)x(\tau(t))$ and the following conditions are satisfied

(A1) $a_1(t),\ a_2(t),\ p\,(t)$ and $q\,(t)\ \in C([t_0,\infty)\,,\ (0,\infty)),\ 0 \le p(t) < 1$;
(A2) $\alpha_1,\ \alpha_2$ are quotient of odd positive integers;
(A3) $f \in C(\mathbb{R},\mathbb{R})$ such that $xf(x) > 0$, $f'(x) > 0$ for all $x \neq 0$ and $-f(-xy) \ge f(xy) \ge f(x)f(y)$ for $xy > 0$;
(A4) $g(t) \in C^1([t_0,\infty)\,,\mathbb{R}),\ g(t) \le t$, for $t \in [t_0,\infty)$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} g(t) = \infty$.

We mean by a solution of equation (1) a function $x\,(t)\ :\ [t_x,\infty)\ \to\ \mathbb{R},\ t_x\ \ge\ t_0$ such that $z\,(t)\,,\ a_1(t)\left(z'(t)\right)^{\alpha_1},\ a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}$ are continuously differentiable for all $t \in [t_x,\infty)$ and satisfies (1) for all $t \in [t_x,\infty)$ and satisfy $\sup\{|x\,(t)| : t \ge T\} > 0$ for any $T \ge t_x$. A solution of equation (1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. In the sequel it will be always assumed that equation (1) has nontrivial solutions which exist for all $t_0 \ge 0$. Equation (1) is called oscillatory if all solutions are oscillatory. In the last few years, the oscillation theory and asymptotic behavior of differential equations and their applications have received more and more attentions, the reader is referred to the papers [1]-[18] and the references cited there in. In fact, Grace *et al* [12] studied the third order nonlinear differential equation of the form

$$\left(a(t)\left(x''(t)\right)^{\alpha}\right)' + q(t)f(x(g(t))) = 0. \tag{2}$$

by comparing equation (2) with a pair of first order delay differential equations. They show that the oscillation of both of these first order equations implies the oscillation of equation (2). Baculikova and Džurina [7] investigate oscillatory behavior of solutions of equation (2), which extended and improved the results given in [12]. Baculikova and Džurina [6] considered the third order nonlinear neutral differential equation of the form

$$[a(t)\{(x(t) + p(t)x(\tau(t)))''\}^{\gamma}]' + q(t)f(x(g(t))) = 0, \tag{3}$$

where $g(t) \leq t$. Our aim is to investigate the oscillatory criteria for all solutions of equation (1) with the case, for $k = 1, 2$

$$\int_{t_0}^{\infty} a_k^{-\frac{1}{\alpha_k}}(t)dt = \infty, \tag{4}$$

By using a Riccati transformation technique and new comparison principles that enable us to deduce properties of the third order nonlinear differential equation from oscillation the first order nonlinear delay differential equation.

2. Main Results

The following lemmas will be needed later.

Lemma 1. *[6, Lemma 2.11] Suppose that $x(t)$ is nonoscillatory solution of (3) such that $\frac{x(t)}{t}$ is bounded. Assume that the corresponding function $z(t)$ satisfies $\lim_{t\to\infty} \frac{z(t)}{t} = l$. If, in addition,*

$$\lim_{t\to\infty} p(t) = p_* \in (0, 1), \tag{5}$$

$$\lim_{t\to\infty} \frac{\tau(t)}{t} = \sigma_* < \infty, \; p_* \, \sigma_* \neq 1, \tag{6}$$

then

$$\lim_{t\to\infty} \frac{x(t)}{t} = \frac{l}{1 + p_* \sigma_*}.$$

Lemma 2. *[6, Lemma 2.12] Let (5) holds. Suppose that $x(t)$ is nonoscillatory solution of (3) such that $\frac{x(t)}{A(t)}$ is bounded. Assume that the corresponding function $z(t)$ satisfies $\lim_{t\to\infty} \frac{z(t)}{A(t)} = l$. If*

$$\lim_{t\to\infty} \frac{A(\tau(t))}{A(t)} = \sigma_0 < \infty, \; p_* \, \sigma_* \neq 1, \tag{7}$$

then

$$\lim_{t\to\infty} \frac{x(t)}{A(t)} = \frac{l}{1 + p_* \sigma_0},$$

where $A(t) := \int_{t_0}^{t} \int_{t_0}^{u} a_2^{-\frac{1}{\alpha_2}}(s)dsdu$.

Before stating our main results, we start with the following lemmas which will play an important role in the proofs of our main results.

Lemma 3. *Assume that (4) holds. Let $x(t)$ be an eventually positive solution of the equation (1). Then there exists a $T \geq t_0$ such that either*

(1) $z'(t) > 0, \; (a_1(t)\,(z'(t))^{\alpha_1})' > 0$ for all $t \geq T$; or (2) $z'(t) < 0, \; (a_1(t)\,(z'(t))^{\alpha_1})' > 0$ for all $t \geq T$.

Proof. Pick $t_1 \geq t_0$ such that $x(g(t)) > 0$, for $t \geq t_1$. From Eq. (1), (A1) and (A3), we have

$$[a_2(t)\{(a_1(t)\,(z'(t))^{\alpha_1})'\}^{\alpha_2}]' \leq 0,$$

for all $t \geq t_1$. That is $a_2(t)\ (a_1(t)\,(z'(t))^{\alpha_1})'$ is strictly decreasing on $[t_1, \infty)$, and thus $z'(t)$ and $(a_1(t)\,(z'(t))^{\alpha_1})'$ are eventually of one sign. We claim that

$$\left(a_1(t)\,(z'(t))^{\alpha_1}\right)' > 0,$$

on $[t_1, \infty)$. If not, then, we have two cases.
Case 1. There exists $t_2 \geq t_1$, sufficiently large, such that

$$z'(t) > 0 \quad \text{and} \quad (a_1(t)\,(z'(t))^{\alpha_1})' < 0 \quad \text{for } t \geq t_2,$$

thus, $a_1(t)\,(z'(t))^{\alpha_1}$is strictly decreasing on $[t_2, \infty)$ and there exists a negative constant M such that $a_2(t)\{(a_1(t)z'(t))^{\alpha_1})'\}^{\alpha_2} \leq M$ for all $t \geq t_2$. Dividing by $a_2(t)$ and integrating from t_2 to t, we obtain

$$a_1(t)(z'(t))^{\alpha_1} \leq a_1(t_2)\,(z'(t_2))^{\alpha_1} + M^{\frac{1}{\alpha_2}} \int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(s)ds.$$

Letting $t \to \infty$ and using (4) then $a_1(t)\,(z'(t))^{\alpha_1} \to -\infty$ which contradicts that $z'(t) > 0$.
Case 2. There exists $t_2 \geq t_1$, sufficiently large, such that

$$z'(t) < 0 \quad \text{and} \quad (a_1(t)\,(z'(t))^{\alpha_1})' < 0 \quad \text{for } t \geq t_2,$$

which implies that

$$a_1(t)\,(z'(t))^{\alpha_1} \leq a_1(t_2)\,(z'(t_2))^{\alpha_1} = k < 0.$$

Dividing by $a_1(t)$ and integrating from t_2 to t, we get

$$z(t) \leq z(t_3) + k^{\frac{1}{\alpha_1}} \int_{t_2}^{t} a_1^{-\frac{1}{\alpha_1}}(s)ds.$$

Letting $t \to \infty$, then (4) yields $z(t) \to -\infty$ this contradicts the fact that $z(t) > 0$. Then, we have $\left(a_1(t)\,(z'(t))^{\alpha_1}\right)' > 0$ for $t \geq t_1$ and of one sign thus either $z'(t) > 0$ or $z'(t) < 0$. □

The next result deals with the case $\tau(t) \leq t$. Define

$$\beta(t,T) := \int_T^t a_1^{-\frac{1}{\alpha_1}}(s) \left[\int_T^s a_2^{-\frac{1}{\alpha_2}}(u)du\right]^{\frac{1}{\alpha_1}} ds,\ \beta_1(t,T) := \int_T^t a_2^{-\frac{1}{\alpha_2}}(u)du.$$

Theorem 1. *Assume that $0 \leq p(t) \leq p < 1$ and (4) hold. If the first order delay equation*

$$y'(t) + q(t)f(y^{\frac{1}{\alpha_1\alpha_2}}(g(t)))f\left((1 - p(g(t)))\right)f\left(\beta(g(t), T)\right) = 0, \tag{8}$$

is oscillatory and

$$\int_{t_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(v) \left[\int_v^{\infty} a_2^{-\frac{1}{\alpha_2}}(u) \left(\int_u^{\infty} q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}} dv = \infty, \tag{9}$$

then every solution of equation (1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Assume (1) has a nonoscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large such that $x(t) > 0$ and $x(g(t)) > 0$ on $[t_1, \infty)$. From equation (1), (A1) and (A3), we have

$$[a_2(t)\{(a_1(t)\,(z'(t))^{\alpha_1})'\}^{\alpha_2}]' \leq 0$$

for all $t \geq t_1$. That is $a_2(t)\ (a_1(t)\,(z'(t))^{\alpha_1})'$ is strictly decreasing on $[t_1,\infty)$ and thus $(a_1(t)\,(z'(t))^{\alpha_1})'$ and $z'(t)$ are eventually of one sign. By Lemma 3, we have one of the following cases, for $t_2 \geq t_1$, is sufficiently large
(1) $z'(t) > 0,\ (a_1(t)\,(z'(t))^{\alpha_1})' > 0,$
(2) $z'(t) < 0,\ (a_1(t)\,(z'(t))^{\alpha_1})' > 0,$
From Case 1, we have, for $t \geq t_2$

$$\begin{aligned} a_1(t)\left(z'(t)\right)^{\alpha_1} &= a_1(t_2)\left(z'(t_2)\right)^{\alpha_1} + \int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(s) y^{\frac{1}{\alpha_2}}(s) ds \\ &\geq y^{\frac{1}{\alpha_2}}(t) \int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(s) ds, \end{aligned}$$

where $y(t) := a_2(t)\{(a_1(t)\,(z'(t))^{\alpha_1})'\}^{\alpha_2}$. It follows that

$$z'(t) \geq a_1^{-\frac{1}{\alpha_1}}(t)\, y^{\frac{1}{\alpha_1\alpha_2}}(t) \left[\int_{t_2}^{t} a_2^{-\frac{1}{\alpha_2}}(s) ds\right]^{\frac{1}{\alpha_1}}. \tag{10}$$

Integrating the above inequality from t_2 to t, we get

$$\begin{aligned} z(t) &\geq \int_{t_2}^{t} a_1^{-\frac{1}{\alpha_1}}(s) y^{\frac{1}{\alpha_1\alpha_2}}(s) \left[\int_{t_2}^{s} a_2^{-\frac{1}{\alpha_2}}(u) du\right]^{\frac{1}{\alpha_1}} ds \\ &\geq y^{\frac{1}{\alpha_1\alpha_2}}(t) \beta(t,t_2). \end{aligned}$$

There exists $t_3 \geq t_2$ with $g(t) \geq t_2$ for all $t \geq t_3$ such that

$$z(g(t)) \geq y^{\frac{1}{\alpha_1\alpha_2}}(g(t))\beta(g(t),t_2). \tag{11}$$

Since $z'(t) > 0$ and $\tau(g(t)) \leq g(t)$, then

$$\begin{aligned} x(g(t)) &= z(g(t)) - p(g(t))x(\tau(g(t))) \\ &\geq z(g(t)) - p(g(t))z(\tau(g(t))) \\ &\geq z(g(t))\left(1 - p(g(t))\right). \end{aligned} \tag{12}$$

The above inequality and (11) yield

$$x(g(t)) \geq \left(1 - p(g(t))\right) y^{\frac{1}{\alpha_1\alpha_2}}(g(t))\beta(g(t),t_2).$$

From equation (1) and (A3), we have

$$\begin{aligned} -y'(t) &= q(t) f(x(g(t))) \\ &\geq q(t) f\left((1 - p(g(t)))\right) f(y^{\frac{1}{\alpha_1\alpha_2}}(g(t))) f\left(\beta(g(t),t_2)\right). \end{aligned}$$

Integrating the above inequality from t to ∞, we get

$$y(t) \geq \int_t^{\infty} q(s) f(y^{\frac{1}{\alpha_1\alpha_2}}(g(s))) f\left((1 - p(g(s))))\right) f\left(\beta(g(s),t_2)\right) ds.$$

The function $y(t)$ is obviously strictly decreasing. Hence, by [18, Theorem 1] there exists a positive solution of equation (8) with $\lim_{t\to\infty} y(t) = 0$ which contradicts that (8) is oscillatory.

For the Case 2. Pick $t_1 \geq t_0$ such that $x(g(t)) > 0$, for $t \geq t_1$. Since $x(t)$ is an eventually positive solution of the equation (1) for all $t \in [t_0, \infty)$ and $z'(t) < 0$, then $\lim_{t\to\infty} z(t) = l_1 \geq 0$. Assume that $l_1 > 0$, then, for any $\epsilon > 0$, we have $l + \epsilon > z(t) > l$, eventually. Choose $0 < \epsilon < \frac{l(1-p)}{p}$, we get

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &\geq l - pz(\tau(t)) \\ &\geq l - p(l+\epsilon) > kz(t) \end{aligned} \tag{13}$$

where, $k := \frac{l-p(l+\epsilon)}{l+\epsilon} > 0$, $z(g(t)) \geq l_1$ for $t \geq t_4 \geq t_3$. Integrating equation (1) from t to ∞, we obtain

$$\begin{aligned} a_2(t)\{(a_1(t)\left(z'(t)\right)^{\alpha_1})'\}^{\alpha_2} &\geq \int_t^\infty q(s)f(x(g(s)))ds \\ &\geq \int_t^\infty q(s)f(kz(g(s)))ds. \end{aligned}$$

It follows from (A3) that

$$(a_1(t)\left(z'(t)\right)^{\alpha_1})' \geq (f(k)f(l))^{\frac{1}{\alpha_2}} a_2^{-\frac{1}{\alpha_2}}(t)\left(\int_t^\infty q(s)ds\right)^{\frac{1}{\alpha_2}},$$

Integrating the above inequality from t to ∞, we get

$$-z'(t) \geq \frac{b}{a_1^{\frac{1}{\alpha_1}}(t)}\left[\int_t^\infty a_2^{-\frac{1}{\alpha_2}}(u)\left(\int_u^\infty q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}}.$$

where $b := (f(k)f(l))^{\frac{1}{\alpha_1\alpha_2}}$. By integrating the last inequality from t_4 to ∞, we have

$$z(t_4) \geq b\int_{t_4}^\infty a_1^{-\frac{1}{\alpha_1}}(v)\left[\int_v^\infty a_2^{-\frac{1}{\alpha_2}}(u)\left(\int_u^\infty q(s)ds\right)^{\frac{1}{\alpha_2}} du\right]^{\frac{1}{\alpha_1}} dv.$$

This contradicts to the condition (9), then $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \leq z(t)$ then $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Remark 1. *When $a_1(t) \equiv 1$ and $\alpha_1 \equiv 1$, Theorem 1 is reduced to [6, Theorem 2. 4].*

Example 1. *Consider the third order delay differential equation*

$$[t\{(\frac{1}{t^2}((x(t)+\frac{1}{2}x(\frac{t}{3})')^{\frac{1}{5}})'\}^5]' + \frac{1}{t}f(x(\frac{t}{2})) = 0,\ t \geq 1. \tag{14}$$

We note that

$$\begin{aligned} &f(y) = y,\ \tau(t) = \frac{t}{3} < t, \\ &g(t) = \frac{t}{2} < t,\ g'(t) > 0,\ \lim_{t\to\infty} g(t) = \lim_{t\to\infty}\frac{t}{2} = \infty, \end{aligned}$$

and

$$a_1(t) = \frac{1}{t^2},\ a_2(t) = t,\ \alpha_1 = \frac{1}{5},\ \alpha_2 = 5,$$

and

$$\int_1^\infty a_1^{-\frac{1}{\alpha_1}}(u)du = \infty,\ \int_1^\infty a_2^{-\frac{1}{\alpha_2}}(u)du = \infty.$$

It easy to see that condition (9) holds and Eq.(8), reduces to

$$y'(t)+\frac{1}{t}\left(b_1+b_2t^{15}-b_3t^{71/5}+b_4t^{67/5}-b_5t^{63/5}+b_6t^{59/5}-b_7t^{11}\right)y\left(\frac{t}{2}\right)=0. \tag{15}$$

where $b_i>0, i=1\to 7$. On the other hand, Theorem 1 guarantees the oscillation of (15). Since

$$\lim_{t\to\infty}\int_{\frac{t}{2}}^{t}\left(\frac{b_1}{s}+b_2s^{14}-b_3s^{56/5}+b_4s^{62/5}-b_5s^{48/5}+b_6s^{54/5}-b_7s^{10}\right)ds>\frac{1}{e}.$$

Then equation (15) is oscillatory and according to Theorem 1 every nonoscillatory solution of Eq.(14) tends to zero as $t\to\infty$.

When $\tau(t)\geq t$, we obtain the following result.

Theorem 2. *Let $g'(t)>0$ on $[t_0,\infty)$ and (4) hold and there exists a function $\xi(t)$ such that*

$$\xi'(t)\geq 0,\ \xi(t)>t \text{ and } \eta(t)=g(\xi(\xi(t)))<t \tag{16}$$

If the first order delay equations

$$z'(t)+q_1(t)f^{\frac{1}{\alpha_1\alpha_2}}(z((\eta(t)))=0, \tag{17}$$

where,

$$q_1(t):=a_1^{-\frac{1}{\alpha_1}}(t)(\int_t^{\xi(t)}a_2^{-\frac{1}{\alpha_2}}(u)(\int_u^{\xi(u)}q(s)f((1-p(g(s))))ds)^{\frac{1}{\alpha_2}}du)^{\frac{1}{\alpha_1}},$$

is oscillatory. Then every solution of (1) is either oscillatory or $\limsup_{t\to\infty}|x(t)|=\infty$.

Proof. Assume (1) has a nonoscillatory solution. Then, without loss of generality, there is a $t_1\geq t_0$, sufficiently large such that $x(t)>0$ and $x(g(t))>0$ on $[t_1,\infty)$. From equation (1), (A1) and (A3), we have

$$[a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}]'\leq 0$$

for all $t\geq t_1$. That is $a_2(t)\ (a_1(t)(z'(t))^{\alpha_1})'$ is strictly decreasing on $[t_1,\infty)$ and thus $(a_1(t)(z'(t))^{\alpha_1})'$ and $z'(t)$ are eventually of one sign. Then, from Lemma 3 we have, one of the following cases

(1) $z'(t)>0,\ (a_1(t)(z'(t))^{\alpha_1})'>0,$

(2) $z'(t)<0,\ (a_1(t)(z'(t))^{\alpha_1})'>0,$

For the Case 1. Since $z(t)>0$ and $z'(t)>0$ then $\lim_{t\to\infty}z(t)=\infty$ and from the definition of $z(t)$, we have $\limsup_{t\to\infty}|x(t)|=\infty$.

For the Case 2. Since $z'(t)<0$ and $\tau(t)\geq t$, we obtain

$$x(t)=z(t)-p(t)x(\tau(t))\geq z(t)-p(t)z(\tau(t))\geq z(t)(1-p(t)).$$

There exists $t_3\geq t_2$ with $g(t)\geq t_2$ for all $t\geq t_3$ such that

$$x(g(t))\geq z(g(t))(1-p(g(t))).$$

Thus equation (1) and (A3) yield

$$\begin{aligned}-[a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}]' &= q(t)f(x(g(t)))\\ &\geq q(t)f((1-p(g(t))))f(z(g(t))).\end{aligned}$$

By integrating the above inequality from t to $\xi(t)$, we get

$$a_2(t)\{(a_1(t)\,(z'(t))^{\alpha_1})'\}^{\alpha_2} \geq f(z(g(\xi(t))))\int_t^{\xi(t)} q(s)f\left((1-p(g(s)))\right)ds,$$

then,

$$(a_1(t)\,(z'(t))^{\alpha_1})' \geq a_2^{-\frac{1}{\alpha_2}}(t)f^{\frac{1}{\alpha_2}}(z(g(\xi(t))))(\int_t^{\xi(t)} q(s)f\left((1-p(g(s)))\right)ds)^{\frac{1}{\alpha_2}}.$$

Again, integrate the above inequality from t to $\xi(t)$, we obtain

$$\begin{aligned}
&-a_1(t)\,(z'(t))^{\alpha_1}\\
&\geq \int_t^{\xi(t)} a_2^{-\frac{1}{\alpha_2}}(u)f^{\frac{1}{\alpha_2}}(z(g(\xi(u))))(\int_u^{\xi(u)} q(s)f\left((1-p(g(s)))\right)ds)^{\frac{1}{\alpha_2}}du\\
&\geq f^{\frac{1}{\alpha_2}}(z(\eta(t)))\int_t^{\xi(t)} a_2^{-\frac{1}{\alpha_2}}(u)(\int_u^{\xi(u)} q(s)f\left((1-p(g(s)))\right)ds)^{\frac{1}{\alpha_2}}du.
\end{aligned}$$

It follows that,

$$-z'(t) \geq q_1(t)f^{\frac{1}{\alpha_1\alpha_2}}(z(\eta(t))).$$

Hence, by [18, Theorem 1] there exists a positive solution of equation (17) with $\lim_{t\to\infty} z(t) = 0$ which contradicts that (8) is oscillatory. □

When $a_1(t) = 1$ and $\alpha_1 = 1$, Theorem 2 is reduced to [6, Theorem 2. 10].

Lemma 4. *Suppose that $x(t)$ is nonoscillatory solution of (1) such that $\frac{x(t)}{t}$ is bounded. Assume that the corresponding function $z(t)$ satisfies $\lim_{t\to\infty} \frac{z(t)}{t} = l$. If (5) and (6) hold then*

$$\lim_{t\to\infty} \frac{x(t)}{t} = \frac{l}{1+p_*\sigma_*}.$$

Lemma 5. *Let (5) holds. Suppose that $x(t)$ is nonoscillatory solution of (1) such that $\frac{x(t)}{\beta(t,T)}$ is bounded. Assume that the corresponding function $z(t)$ satisfies $\lim_{t\to\infty} \frac{z(t)}{\beta(t,T)} = l$. If*

$$\lim_{t\to\infty} \frac{\beta(\tau(t),T)}{\beta(t,T)} = \sigma_0 < \infty,\ p_*\sigma_* \neq 1, \tag{18}$$

then

$$\lim_{t\to\infty} \frac{x(t)}{\beta(t,T)} = \frac{l}{1+p_*\sigma_0}.$$

Theorem 3. *Let (5)-(7) hold. Assume that all condition of Theorem 2 are satisfied. Then every nonoscillatory solution $x(t)$ of (1) satisfies one of the conditions:*

$$\lim_{t\to\infty} \frac{t^{\epsilon}\,|x(t)|}{\beta(t,T)} = \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{|x(t)|}{\beta(t,T)} = k_2 > 0, \tag{19}$$

$$\lim_{t\to\infty} t^{\epsilon-1}\,|x(t)| = \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{|x(t)|}{t} = k_1 > 0, \tag{20}$$

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} = \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{|x(t)|}{\beta(t,T)} = 0, \tag{21}$$

where $\epsilon > 0$ is arbitrary.

Proof. Assume that $x(t) > 0$ for all $t \in [t_0,\infty)$. From Theorem 2 and (1), we have

$$z(t) > 0,\ z'(t) > 0,\ (a_1(t)\,(z'(t))^{\alpha_1})' > 0,\ [a_2(t)\{(a_1(t)\,(z'(t))^{\alpha_1})'\}^{\alpha_2}]' \leq 0.$$

It follows that

$$\lim_{t\to\infty} a_2(t)\{(a_1(t)\,(z'(t))^{\alpha_1})'\}^{\alpha_2} = l_1.$$

Using l'Hospital's rule, we get

$$\lim_{t\to\infty} \frac{z(t)}{\beta(t,T)} = \left(\lim_{t\to\infty} \frac{a_1(t)\,(z'(t))^{\alpha_1}}{\beta_1(t,T)}\right)^{\frac{1}{\alpha_1}} = \left(\lim_{t\to\infty} a_2(t)\{(a_1(t)\,(z'(t))^{\alpha_1})'\}^{\alpha_2}\right)^{\frac{1}{\alpha_1\alpha_2}} = l,$$

where $l := l_1^{\frac{1}{\alpha_1\alpha_2}}$. Since $0 < \frac{x(t)}{\beta(t,T)} < \frac{z(t)}{\beta(t,T)}$, then $\frac{x(t)}{\beta(t,T)}$ is bounded. From Lemma 5, we get

$$\lim_{t\to\infty} \frac{x(t)}{\beta(t,T)} = \frac{l}{1+p_*\sigma_0} = k_2.$$

Therefore, if $l \neq 0$, then $x(t)$ satisfies (19). If $l = 0$, then $\lim_{t\to\infty} \frac{x(t)}{\beta(t,T)} = 0$. Let

$$\lim_{t\to\infty} z'(t) = l_1.$$

By l'Hospital's rule, we obtain

$$\lim_{t\to\infty} \frac{z(t)}{t} = \lim_{t\to\infty} z'(t) = l_1.$$

If $l_1 < \infty$, then by Lemma 4, we get

$$\lim_{t\to\infty} \frac{x(t)}{\beta(t,T)} = \frac{l_1}{1+p_*\sigma_0} = k_1.$$

That is $x(t)$ satisfies (20). If $l_1 = \infty$, then by Lemma 4, we have

$$\limsup_{t\to\infty} \frac{x(t)}{t} = \infty.$$

Then$x(t)$ satisfies (21). $\square$

Example 2. *Consider the third order delay differential equation*

$$[\frac{1}{t^2}\{(\frac{1}{t}((x(t)+\frac{1}{4}x(3t)')^3)'\}^{1/3}]' + \frac{1}{t^3}f(x(\frac{t}{4})) = 0,\ t \geq 1. \tag{22}$$

We note that

$$f(y) = y,\ \tau(t) = 3t > t,\ \xi(t) = \frac{3}{2}t > t,$$
$$g(t) = \frac{t}{4} < t,\ g'(t) > 0,\ \lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{t}{4} = \infty,$$

and

$$a_1(t) = \frac{1}{t},\ a_2(t) = \frac{1}{t^2},\ \alpha_1 = 3,\ \alpha_2 = 1/3,$$

and

$$\int_1^\infty a_1^{-\frac{1}{\alpha_1}}(u)du = \infty,\ \int_1^\infty a_2^{-\frac{1}{\alpha_2}}(u)du = \infty.$$

It is easy to see that Eq.(17) *is reduced to*

$$y'(t) + ct^{\frac{5}{3}}y\left(\frac{9t}{16}\right) = 0. \tag{23}$$

where $c > 0$. *Since*

$$\lim_{t\to\infty}\int_{\frac{t}{4}}^{t}\left(cs^{\frac{5}{3}}\right)ds > \frac{1}{e},$$

then (23) *is oscillatory according to* [17, *Theorem* 2.1.1]. *By Theorem* 3, *every nonoscillatory of* (22) *satisfies one of the conditions* (19)- (21)

Now, when $\tau(t) \geq t$, we obtain the following result using the Riccati transformation techniques .

Theorem 4. *Let* $g'(t) > 0$ *on* $[t_0, \infty)$, $\alpha_1\alpha_2 = 1$, $\frac{f(u)}{u} \geq K > 0$, (4) *and* (9) *hold. Furthermore, assume that there exists a positive function* ρ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(q(s)\left(1-p(g(s))\right)-\frac{\rho'^2(s)}{4\rho(s)g'(s)\beta_1(g(s),T)}\right)ds = \infty \tag{24}$$

where $\beta_1(g(t),T) := a_1^{-\frac{1}{\alpha_1}}(g(t))\left[\int_T^{g(t)} a_2^{-\frac{1}{\alpha_2}}(s)ds\right]^{\frac{1}{\alpha_1}}$, *for* $g(t) \geq T$. *Then every solution of equation* (1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. To the contrary assume that equation (1) has a nonoscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$, sufficiently large such that $x(t) > 0$ and $x(g(t)) > 0$ on $[t_1, \infty)$. From (1), (A1) and (A3), we have $[a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}]' \leq 0$ for all $t \geq t_1$. Then $a_2(t)\,(a_1(t)(z'(t))^{\alpha_1})'$ is strictly decreasing on $[t_1, \infty)$. That is $(a_1(t)(z'(t))^{\alpha_1})'$ and $z'(t)$ are eventually of one sign. By Lemma 3 we have, one of the following two cases, for $t_2 \geq t_1$, is sufficiently large
(1) $z'(t) > 0,\ (a_1(t)(z'(t))^{\alpha_1})' > 0$,
(2) $z'(t) < 0,\ (a_1(t)(z'(t))^{\alpha_1})' > 0$,
Case 1. In this case, define the function $w(t)$ by

$$w(t) := \rho(t)\frac{a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}}{z(g(t))}.$$

Then

$$\begin{aligned} w'(t) = {} & \frac{\rho(t)}{z(g(t))}[a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}]' \\ & - a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}\frac{\rho(t)z'(g(t))}{z^2(g(t))}g'(t) \\ & + \frac{a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}}{z(g(t))}\rho'(t). \end{aligned}$$

It follows from equation (1) that

$$\begin{aligned} w'(t) = {} & \frac{\rho(t)}{z(g(t))}[-q(t)f(x(g(t)))] \\ & - a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}\frac{\rho(t)z'(g(t))}{z^2(g(t))}g'(t) \\ & + \frac{a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}}{z(g(t))}\rho'(t). \end{aligned}$$

From (10) there exists $t_3 \geq t_2$ with $g(t) \geq t_2$ for all $t \geq t_3$ such that

$$z'(g(t)) \geq \beta_1(g(t),t_2)y^{\frac{1}{\alpha_1\alpha_2}}(g(t)),$$

where $y(t) := a_2(t)\{(a_1(t)(z'(t))^{\alpha_1})'\}^{\alpha_2}$. Since $y' < 0,\ g(t) < t$, we get

$$z'(g(t)) \geq \beta_1(g(t),t_2)y^{\frac{1}{\alpha_1\alpha_2}}(t).$$

From the above inequality and (12), we obtain

$$w'(t) \le -Kq(t)\,(1-p(g(t))) + \frac{y(t)}{z(g(t))}\rho'(t) - \frac{1}{z^2(g(t))}y^2(t)\beta_1(g(t),t_2)\rho(t)g'(t).$$

Thus

$$\begin{aligned} w'(t) &\le -Kq(t)\,(1-p(g(t))) + \frac{\rho'(t)}{\rho(t)}w(t) - g'(t)\beta_1(g(t),t_2)\frac{w^2(t)}{\rho(t)}, \\ &= -Kq(t)\,(1-p(g(t))) + \frac{\rho'^2(t)}{4\rho(t)g'(t)\beta_1(g(t),t_2)} \\ &- \left(\sqrt{\frac{g'(t)\beta_1(g(t),t_2)}{\rho(t)}}w(t) - \frac{\rho'(t)}{2\sqrt{\rho(t)g'(t)\beta_1(g(t),t_2)}}\right)^2. \end{aligned}$$

and hence

$$w'(t) \le -Kq(t)\,(1-p(g(t))) + \frac{\rho'^2(t)}{4\rho(t)g'(t)\beta_1(g(t),t_2)}.$$

Integrate the above inequality from t_2 to t, we have

$$w(t) \le w(t_2) - \int_{t_2}^{t}\left(Kq(s)\,(1-p(g(s))) - \frac{\rho'^2(s)}{4\rho(s)g'(s)\beta_1(g(s),t_2)}\right)ds.$$

Letting $t \to \infty$ and using (24), we get $w(t) \to -\infty$ which contradicts that $w(t) > 0$. □

Consider the third order delay differential equation

$$[t\{(\frac{1}{t}((x(t)+\frac{1}{2}x(3t)')^3)'\}^{1/3}]' + t^3 f(x(\frac{t}{2})) = 0,\ t \ge 1. \tag{25}$$

We note that

$$\begin{aligned} &f(y) = y,\ \tau(t) = 3t > t,\ \rho(t) = 1, \\ &g(t) = \frac{t}{2} < t,\ g'(t) > 0,\ \lim_{t\to\infty} g(t) = \lim_{t\to\infty}\frac{t}{2} = \infty. \end{aligned}$$

Also,

$$a_1(t) = \frac{1}{t},\ a_2(t) = t,\ \alpha_1 = 1/3,\ \alpha_2 = 3,$$

and

$$\int_1^\infty a_1^{-\frac{1}{\alpha_1}}(u)du = \infty,\ \int_1^\infty a_2^{-\frac{1}{\alpha_2}}(u)du = \infty.$$

It is easy to see that (9) and (24) are hold. Then every nonoscillatory solution of equation (25) tends to zero as $t \to \infty$.

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References

1. R. P. Agarwal, S. R. Grace and D. O'Regan, Oscillation Theory for difference and functional Differential Equations, Kluwer, Dordrecht, **2000**.
2. R. P. Agarwal, S. R. Grace and D. O'Regan, Oscillation Theory for second order Dynamic Equations, Taylor & Francis, London, **2003**
3. R. P. Agarwal, S. R. Grace and D. O'Regan, on the oscillation of certain functional differential equations via comparison methods, *J. Math. Anal. Appl.* **2003**, *286*, 577–600.
4. R. P. Agarwal, S. R. Grace and D. O'Regan, The oscillation of certain higher order functional differential equations, *Adv. math. comput. Modell.* **2003**, *37*, 705–728.

5. R. P. Agarwal, S. R. Grace and T. Smith, Oscillation of certain third order functional differential equations, *Adv. Math. Sci. Appl.* **2006**, *16*, 69-94.
6. B. Baculikova and J. Džurina, Oscillation of third-order functional differential equations, *E. J.. Qualitative Theory of Diff. Equ.* **2010**, *43* 1–10.
7. B. Baculikova and J. Džurina, Oscillation of third-order nonlinear differential equations, *Appl. Math. Letters* **2011**, *24* 466–470.
8. B. Baculikova, E. M. Elabbasy, S. H. Saker, and J. Džurina,Oscillation criteria for third- order nonlinear differential equations, *Math. Slovaca* **2008**, *58*, 201–220.
9. **T. A. Chanturia,** On some asymptotic properties of solutions of ordinary differential equations, Dok1. Akad. Nauk SSSR, 235 (1977), No5.: Soviet Math. dok1. 18, No4 (1977), 1123-1126.
10. L. H. Erbe, T. S. Hassan and A. Peterson, Oscillation of third order nonlinear functional dynamic equations on time scales, *Differential equations and Dynamical Systems* **2010**,*18*, 199–227.
11. L. H. Erbe, Q. Kong and B. Z. Zhan, Oscillation theory for functional differential equations, *Marcel Dekker, New York* **1995**.
12. S. R. Grace, R. P. Agarwal, R. Pavani and E.Thandapani, on the oscilation certain third order nonlinear functional differential equations, *Appl. Math. Comput.* **2008**, *202*, 102–112. Zbl 1154.34368.
13. I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations With Applications, *Clarendon Press, Oxford* **1991**.
14. T. S. Hassan, Oscillation of third order nonlinear delay dynamic equations on time scales, *Math. comput. Modelling.* **2009**, *49*, 1573–1586.
15. Y. Kitamura, Oscillation of functional differential equations with general deviating arguments, *Hiroshima Math. J.* **1985**, *15*, 445–491.
16. T. Kusano, and B. S. Lalli, On oscillation of half -linear functional differential equations with deviating arguments, *Hiroshima Math. J.* **1994**, *24*, 549–563.
17. G. S. Ladde,V. **l**akshmikantham and B. G. Zhang, Oscillation Theory of Differential Equations With Deviating Arguments, *Macel Dekker, New York*, **1987**.
18. Ch. G. Philos, On the nonoscillatory solutions tending to zero at ∞ to differential equations with positive delays, *Arch. Math. 36* **1981**, *36*, 168–178.

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